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A BLOWUP RESULT FOR THE CRITICAL EXPONENT IN CONES

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ABSTRACT

We consider positive solutions of the initial value problem for $u_t = \Delta u + u^p$ in cones $D = \mathbf{R}^+ \times \Omega \subseteq \mathbf{R}^N$ $(\Omega \subseteq S^{N-1})$. In an earlier paper, we determined a critical exponent $p^*(\Omega)$ with the following properties: (a) if $1 < p < p^*$, then all nontrivial solutions blow up in finite time (blowup case); (b) if $p > p^*$, then there are nontrivial global solutions (global existence case). Here we show that p^* belongs to the blowup case. This generalizes a well-known result for the critical exponent $p^* = 1 + 2/N$ in $D = \mathbf{R}^N$.

Let $D \subset R^N$ be an unbounded domain. We consider the initial-boundary value problem

$$(\mathrm{P})\qquad \begin{aligned} &u_t = \Delta u + u^p && \text{in } D \times [0, T),\\ &u(x,t) = 0 && \text{on } \partial D \times [0, T),\\ &u(x,0) = u_0(x) && \text{on } D,\end{aligned}$$

$$u \text{ is bounded at } |x| = \infty,$$

where $u_0 \geqq 0$ and $p > 1$.

In the case $D = R^N$, a classical result of Fujita [3] says:

(A) If $1 < p < 1 + 2/N$, there are no nontrivial nonnegative solutions of (P).

(B) If $p > 1 + 2/N$, global, positive solutions of (P) exist. (That is, if

$$0 \leqq u_0(x) < \delta(4\pi t_0)^{-N/2}\exp(-|x|^2/4t_0)$$

for some $t_0 > 0$ and some $\delta = \delta(t_0)$, sufficiently small, then u is global.)

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Case (A) is called the blowup case; (B) is called the global existence case. The number

$$p^* = 1 + 2/N$$

is called the critical exponent. Several authors [1, 5, 7, 14] have shown that p^* belongs to the blow up case.

Fujita asked the following question: If R^N is replaced by the exterior of a bounded domain, is p^* still the critical exponent? (The answer is yes and this has been established in [2].) The question then arises, if both the domain D and $R^N - D$ are unbounded, what happens to p^*? Meier [12] gave a partial answer when, for fixed $k \in \{1, \ldots, N\}$,

$$D = D_k = \{x \in R^N \mid x_1 > 0, \ldots, x_k > 0\}.$$

He found that $p^* = 1 + 2/(N + k)$. He did *not* prove that p^* belongs to the blow up case.

Meier's result in turn led the authors of [2] to consider general cones. We turn next to a brief discussion of the results for cones.

By a cone D in R^N with vertex at 0, we mean the following: Let $\Omega \subset S^{N-1}$ be an open connected subset of the unit N sphere, then D has the form

$$\{x \in R^N \mid x \neq 0, x/|x| \in \Omega\}.$$

We let $r = |x|$. For any $x \in D$, we write $x = (r, \theta)$ in "polar" coordinates.

Let ω_1 be the smallest Dirichlet eigenvalue for the Laplace Beltrami operator Δ_θ on Ω. Let $\gamma_\pm$ denote the positive and negative roots of

$$\gamma(\gamma + N - 2) - \omega_1 = 0.$$

Let

$$p^*(\omega_1) = 1 + 2/(N + \gamma_+) = 1 + 2/(-\gamma_-).$$

In [2] it was shown that if $1 < p < p^*(\omega_1)$, p is in the blow up case (A). There the authors also showed that if p was sufficiently large, p was in the global existence case. In [10], this result was sharpened and it was shown that if $p > p^*(\omega_1)$, then p is in the global existence case (B).

It is the purpose of this paper to show that $p^*(\omega_1)$ is also in the blow up case (Theorem 3 below).

We will do this by modifying the arguments of Weissler [14] for the case $D = R^N$. Weissler's proof made strong use of the fact that the L^1 norm of the Green's function for the heat equation in R^N is independent of time. This is not true for other domains and therefore his arguments must be substantially modified.

This result has also been established by Kavian [6] but only in the case of *convex* cones (cones for which Ω is a convex subset of S^{N-1} in the geodesic metric).

Let $\{\psi_n(\theta)\}_{n=1}^{\infty}$ denote an orthogonal sequence of Dirichlet eigenfunctions for Δ_θ on Ω corresponding to the sequence $\{\omega_n\}$ of Dirichlet eigenvalues for this problem. We shall normalize ψ_1 so that

$$\int_\Omega \psi_1(\theta)dS_\theta = 1.$$

(We may always take $\psi_1 > 0$ in Ω since Δ_θ (with Dirichlet boundary conditions) is a self adjoint second order elliptic operator on $H_0^1(\Omega)$. See Courant and Hilbert, *Methods of Mathematical Physics*, Vol. I, p. 452.)

Throughout this paper, computable constants C or C_i, $i = 0, 1, 2, \ldots$, depend upon ω_1, ψ_1, N. This dependence is not explicitly indicated. When they depend upon other variables, we indicate that dependence in the argument list.

Define

$$\nu_n = [\tfrac{1}{4}(N-2)^2 + \omega_n]^{1/2}$$

for $n = 1, 2, 3, \ldots$. Then the Green's function for the linear heat equation in the cone takes the following form for some appropriate sequence $\{c_n\}_{n=1}^{\infty}$ of positive constants:

$$G(r, \theta, \rho, \phi; t) = (2t)^{-1}(r\rho)^{-(N-2)/2}\exp\left(-\frac{(\rho^2+r^2)}{4t}\right)\sum_{n=1}^{\infty} c_n I_{\nu_n}\left(\frac{r\rho}{2t}\right)\psi_n(\theta)\psi_n(\phi)$$

where $\phi, \theta \in \Omega$ and

$$I_\nu(t) = (\tfrac{1}{2}z)^\nu \sum_{k=0}^{\infty} \frac{(\tfrac{1}{2}z)^k}{k!\,\Gamma(\nu+k+1)} = \begin{cases} (\tfrac{1}{2}z)^\nu/\Gamma(\nu+1) & z \to 0^+, \\ e^z/\sqrt{2\pi z} & z \to +\infty. \end{cases}$$

denotes the usual modified Bessel function. The formula for G can be obtained by expanding the inverse Laplace transform of the solution of the heat equation in a Fourier–Bessel series and using the identity

$$\int_0^\infty e^{-\lambda t} J_{\nu_n}(\sqrt{\lambda}r)J_{\nu_n}(\sqrt{\lambda}\rho)d\lambda = \frac{1}{t}\exp\left(-\frac{r^2+\rho^2}{4t}\right)\cdot I_\nu\left(\frac{r\rho}{2t}\right) \qquad [13].$$

Then we have the following inequality for G:

$$\int_{\Omega} G(r,\theta,\rho,\phi,t)\psi_1(\theta)dS_\theta$$

$$= C_0(2t)^{-1}(r\rho)^{-(N-2)/2} I_\nu\left(\frac{r\rho}{2t}\right) e^{-(r^2+\rho^2)/4t}\psi_1(\phi)\int_{\Omega}\psi_1^2\,dS_\theta \tag{E1}$$

$$\geqq Ct^{-(\gamma+N/2)}(r\rho)^\gamma e^{-(r^2+\rho^2)/4t}\psi_1(\phi)$$

where we have set $\gamma=\gamma_+$ and $\nu=\nu_1=\gamma+\frac{1}{2}(N-2)$ and where C is a computable constant. The second line follows from the series representation for $I_\nu(z)$.

Now let $w(r,\theta,t)$ be the solution of the linear heat equation, $w_t=\Delta w$, with the same initial and boundary values as u. Then, by "variation of parameters",

$$u(r,\theta,t)=w(r,\theta,t)+\int_0^t\int_D G(r,\theta,\rho,\phi;t-\eta)u^p(\rho,\phi,\eta)\rho^{N-1}d\rho dS_\phi d\eta$$

where

$$w(r,\theta,t)=\int_D G(r,\theta,\rho,\phi,t)u_0(\rho,\phi)\rho^{N-1}dS_\phi d\rho.$$

Consequently $u\geqq w$ and, by (E1), we have, after changing the order of integration,

$$\int_{\Omega} u(r,\theta,t)\psi_1(\theta)dS_\theta$$

$$\geqq \int_{\Omega} w(r,\theta,t)\psi_1(\theta)dS_\theta \tag{E2}$$

$$\geqq Ct^{-(\gamma+N)/2}r^\gamma e^{-r^2/4t}\int_0^\infty\int_{\Omega}\rho^{\gamma+N-1}e^{(-\rho^2/4t)}u_0(\rho,\phi)\psi_1(\phi)dS_\phi d\rho.$$

Finally, for any nonnegative function $a(r)$, we have

$$\| G(\cdot,\cdot,\rho_0,\phi_0;t_0)a(\cdot)\psi_1(\cdot)\|_{L^1(D)}$$

$$=\int_D G(r,\theta;\rho_0,\phi_0;t_0)a(r)\psi_1(\theta)r^{N-1}dS_\theta dr \tag{E3}$$

$$\geqq Ct_0^{-(\gamma+N/2)}\rho_0^\gamma e^{(-\rho_0^2/4t_0)}\cdot\psi_1(\phi_0)\int_0^\infty a(r)r^{\gamma+N-1}e^{-r^2/4t_0}dr.$$

When $a(r)=r^\alpha$, this last integral is simply

$$\tfrac{1}{2}\Gamma\left(\frac{\alpha+\gamma+N}{2}\right)(4t_0)^{(\alpha+\gamma+N)/2}$$

so that

$$\| G(\cdot,\cdot,\rho_0,\phi_0;t_0)a(\cdot)\psi_1(\cdot)\|_{L^1(D)} \geqq C(\alpha)t_0^{-(\alpha-\gamma)/2}\rho_0^{\gamma}e^{-\rho_0^2/4t_0}\psi_1(\phi_0).$$

We now establish the following Lemmas:

LEMMA 1. *For all $t\in(0,T)$ where T is the length of the existence interval ($T\leqq\infty$), we have*

$$w(r,\theta,t)\leqq((p-1)t)^{-1/(p-1)}.$$

PROOF.

$$\underline{u}(r,\theta,t)=\{w^{-(p-1)}(r,\theta,t)-t(p-1)\}^{-1/(p-1)}$$

is seen to be a subsolution of (P) and consequently cannot blow up before u does.

LEMMA 2. *If u is a global solution of* (P), *then*

$$\int_D e^{-r^2/4t}r^{\gamma}u_0(r,\theta)\psi_1(\theta)dx\leqq C(p)t^{((N+\gamma)/2-1/(p-1))}$$

for some constant $C(p)$.

Suppose $p<p^*$. Then letting $t\to\infty$, we see from this lemma that $u_0\equiv 0$. We thus recover the result of [2] to which we alluded in the introduction.

To prove the lemma, we see that from Lemma 1,

$$(p-1)^{-1/(p-1)}\geqq t^{1/(p-1)}w(\sqrt{t},\theta,t).$$

Integrating over Ω with respect to $\psi_1(\theta)dS_\theta$, we see from (E2) that

$$(p-1)^{-1/(p-1)}$$
$$\geqq Ct^{1/(p-1)}t^{-(\gamma+N)/2}e^{-1/4}\int_0^\infty\int_\Omega \rho^{\gamma+N-1}e^{(-\rho^2/4t)}\psi_1(\phi)u_0(\rho,\phi)d\rho dS_\phi.$$

This implies the result.

REMARK. If u is a global solution, we may replace $u_0(\rho,\theta)$ by $u(\rho,\theta,t_0)$ for any $t_0>0$ in view of the autonomous nature of (P).

THEOREM 3. *If $p=1+2/(N+\gamma)=1-2/\gamma_-$, the problem* (P) *possesses no nontrivial, nonnegative global solution.*

PROOF. Let $p=1+2/(N+\gamma)$. We have, if u is global, for all $t>0$, from Lemma 2 and the variation of parameters formula,

$$C(p) \geqq \| r^{\gamma} u(r, \theta, t) \psi_1(\theta) \|_{L^1(D)}$$
$$\geqq \left\| \left(\int_0^t \int_D r^{\gamma} G(r,\theta,\rho,\phi; t-s)(u(\rho,\phi,s))^p \rho^{n-1} d\rho dS_\phi ds \right) \psi_1(\theta) \right\|_{L^1(D)}$$
$$= \int_0^t \int_0^\infty \int_\Omega (u(\rho,\phi,s))^p$$
$$\times \left(\int_0^\infty \int_\Omega r^{\gamma} \psi_1(\theta) G(r,\theta,\rho,\phi; t-s) r^{N-1} dr dS_\theta \right) \rho^{N-1} d\rho dS_\theta ds.$$

Thus, from (E3)

$$C(p) \geqq C_1 \int_0^t \int_0^\infty \int_\Omega (u(\rho,\phi,s))^p e^{-\rho^2/4(t-s)} \psi_1(\phi) \rho^{N+\gamma-1} d\rho dS_\phi ds.$$

From Jensen's inequality, we have

$$C(p) \geqq C_1 \int_0^t \int_0^\infty \left(\int_\Omega u(\rho,\phi,s) \psi_1(\phi) dS_\phi \right)^p e^{-\rho^2/4(t-s)} \rho^{N+\gamma-1} d\rho ds,$$

since $\int_\Omega \psi_1 dS_\phi = 1$. Thus, raising both sides of (E2) to the pth power we find that

$$C(p) \geqq C_2(p) \int_0^t \int_0^\infty C_3^p(s, u_0)[s^{-(\gamma+N/2)} \rho^{\gamma} e^{-\rho^2/4s}]^p \rho^{\gamma+N-1} e^{-\rho^2/4(t-s)} d\rho ds$$

where

$$C_3(s,\gamma,N,u_0,\psi_1) = \int_0^\infty \int_\Omega R^{\gamma+N-1} e^{-R^2/4s} u_0(R,\phi) \psi_1(\phi) dS_\phi dR$$
$$\geqq C_4(t_0, u_0)$$

provided $0 < t_0 \leqq s$, where $C_4 > 0$ for nontrivial initial data u_0. Thus, for $t \geqq t_0$, there is C_5 such that

$$C_5(p, u_0, t_0) \geqq \int_{t_0}^t \left(\int_0^\infty \rho^{\gamma(p+1)+N-1} e^{-\rho^2(p(t-s)+s)/4s(t-s)} d\rho \right) s^{-p(\gamma+N/2)} ds.$$

If we fix $\delta \in (0,1)$ and assume $t_0 \leqq s \leqq (1-\delta)t$, then $\delta t/(t-s) \leqq 1$ and $s/(t-s) \leqq 1/\delta - 1$. Therefore

$$\frac{s(t-s)}{p(t-s)+s} = \frac{s}{p + s/(t-s)} \geqq \frac{\delta s}{\delta(p-1)+1}.$$

Consequently, there is C_6 such that

$$C_6(p, u_0, t_0, \delta) \geqq \int_{t_0}^{(1-\delta)t} s^{-p(\gamma+N/2)+(\gamma(p+1)+N)/2} ds$$

in view of the change of variables $\sigma = [\delta/(\delta(p-1)+1)]^{1/2} s^{1/2} \rho$. However,

$$- p(\gamma + N/2) + \tfrac{1}{2}(\gamma(p+1) + N) = -\tfrac{1}{2}\gamma p + \tfrac{1}{2}\gamma - \frac{(p-1)N}{2}$$

$$= -\frac{(p-1)}{2}(N+\gamma).$$

Thus,

$$C_6 \geqq \int_{t_0}^{(1-\delta)t} s^{-(p-1)(N+\gamma)/2} ds \to +\infty$$

as $t \to +\infty$ as long as $\frac{1}{2}(p-1)(N+\gamma) \leqq 1$. This is the desired contradiction.

REMARK. If u^p is replaced by $h(t)u^p$ where $h(t) \geqq 0$ and $h(t) \sim t^q$ $(q > -1)$ as $t \to \infty$, the above proof may be modified to show that there are no global (nontrivial) solutions if

$$1 < p \leqq 1 + (2 + 2q)/(N + \gamma).$$

In this case, the right hand side in Lemma 1 is replaced by $[(p-1)\int_0^\infty h(s)ds]^{-1/(p-1)}$ and the right hand side in Lemma 2 by $C(p,h)\cdot t^{(N+\gamma)/2}\cdot(\int_0^t h(s)ds)^{-1/(p-1)}$. Thus, as a first step in the proof of Theorem 3, we have the inequality

$$C(p,h) \geqq \left\| \left(\int_0^t \int_D r^\gamma G(r,\theta,\rho,\phi,t-s) h(s)(u(\rho,\phi,s))^p \rho^{N-1} d\rho dS_\phi ds \right) \psi_1(\theta) \right\|_{L^1(D)}.$$

By the same arguments as above, we obtain a contradiction by concluding that

$$C_6(p, u_0, t_0, h) \geqq \int_{t_0}^{(1-\delta)t} h(s)\cdot s^{-(p-1)(N+\gamma)/2} ds.$$

We were unable to show that if u^p is replaced by $r^\sigma u^p$, $\sigma \geqq 0$, then

$$p^* = 1 + \frac{2+\sigma}{N+\gamma}$$

belongs to the blowup case. We have already shown, [10], that this is the critical exponent in this case.

REMARK. In [2], it was shown that if D is the exterior of a bounded domain, then the critical exponent $p^* = 1 + 2/N$ as in the case considered by Fujita. However, it has yet to be shown that p^* belongs to the blowup case as it does in the case $D = R^N$.

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